Blow-up Phenomena for a Nonlinear Schrödinger Equation

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Non-*linear* Schrödinger equation:

$$\begin{cases} \boldsymbol{i}\partial_t\psi = -\Delta\psi - |\psi|^{q-2}\psi, & (t,x) \in [0,T[\times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), & u_0 : \mathbb{R}^N \to \mathbb{C}, \end{cases} \tag{NLS}$$

where

- $\psi : [0,T[\times \mathbb{R}^N \to \mathbb{C}$;
- $\boldsymbol{i}^2 = -1$;
- $\partial_t\psi$ is the derivative with respect to the time variable;
- $\Delta = \sum_{1\leq i\leq N} \partial_{x_i}^2$ is the Laplacian on $\mathbb{R}^N$;
- $q > 2$ is a real parameter.

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Conservation laws

Formally, *the L^2 norm (the mass)*

$$\|\psi(t,\cdot)\|_{L^2} := \left(\int_{\mathbb{R}^N} |\psi(t,x)|^2 \,\mathrm{d}x\right)^{1/2}$$

and *the energy*

$$\mathcal{E}(\psi(t,\cdot)) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi(t,x)|^2 \,\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} |\psi(t,x)|^q \,\mathrm{d}x$$

where

$$\nabla := (\partial_{x_1}, \ldots, \partial_{x_N}).$$

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Natural space associated to the equation?

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Natural space associated to the equation?

Sobolev space H^1

Definition (Sobolev space H^1)

$$H^1(\mathbb{R}^N;\mathbb{C}) := \left\{ v \in L^2(\mathbb{R}^N;\mathbb{C}) \,\middle|\, \nabla v \in L^2(\mathbb{R}^N;\mathbb{C})^N \right\}$$

- For the L^2 mass: if $v \in H^1(\mathbb{R}^N)$ then v belongs to $L^2(\mathbb{R}^N)$.
- For the energy

$$\mathcal{E}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} |v|^q \,\mathrm{d}x,$$

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Sobolev embedding

Theorem (Sobolev embedding for H^1)

The space $H^1(\mathbb{R}^N;\mathbb{C})$ is embedded in $L^p(\mathbb{R}^N;\mathbb{C})$ for all $p \in [2, 2^*[$ where

$$2^* := \begin{cases} 2N/(N-2) & \text{si } N \geq 3, \\ \infty & \text{si } N \in \{1,2\} \end{cases}$$

is the critical Sobolev exponent.

Conclusion: if $2 < q < 2^*$, the energy

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Well-posedness and blow-up

Theorem (J. Ginibre, G. Velo 1977)

For every initial condition $\psi_0 \in H^1(\mathbb{R}^N;\mathbb{C})$ *and every* $q \in]2, 2^*[$, *there exists a time* $T_{\max} \in]0, +\infty]$ *and a unique continuous solution*

$$\psi : [0, T_{\max}[\to H^1(\mathbb{R}^N;\mathbb{C}), t \mapsto u(t,\cdot)$$

to the nonlinear Schrödinger equation:

$$\boldsymbol{i}\partial_t\psi = -\Delta\psi - |\psi|^{q-2}\psi, \qquad (t,x) \in [0, T_{\max}[\times \mathbb{R}^N.$$

Moreover, the mass and energy conservation laws are satisfied.

If $T_{\max} < +\infty$, there is *finite-time blowup*:

$$\lim_{t \to T_{\max}} \|\nabla u(t,\cdot)\|_{L^2} = +\infty.$$

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Existence of blow-up: Glassey's argument

If $\psi_0 \in H^1(\mathbb{R}^N;\mathbb{C})$ is such that $x\psi_0 \in L^2(\mathbb{R}^N;\mathbb{C})$, then the *variance of* $|\psi(t,x)|^2$

$$V(t) := \int_{\mathbb{R}^N} |x|^2 |\psi(t,x)|^2 \,\mathrm{d}x$$

is well-defined for all $t \in [0, T_{\max}[$.

Integration by parts shows that

$$\partial_{tt} V(t) = 16\mathcal{E}(\psi_0) - \frac{4(N(q-2)-4)}{q} \|\psi\|_{L^q}^q.$$

Therefore, if $q \geq 2 + \frac{4}{N}$, we obtain

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If $q \geq 2 + \frac{4}{N}$, $\psi_0 \in H^1(\mathbb{R}^N; \mathbb{C})$ is such that $x\psi_0 \in L^2(\mathbb{R}^N; \mathbb{C})$ and $\mathcal{E}(\psi_0) < 0$, then the corresponding solution $\psi(t, x)$ of (NLS) blows up in finite time.

Proof.

Under the assumptions of the theorem, the function

$$[0, T_{\max}[\to [0, +\infty[: t \mapsto V(t)$$

is nonnegative and satisfies $\partial_{tt} V(t) \leq E(\psi_0) < 0$. □

R. T. Glassey. "On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations". In: *J. Math. Phys.* 18.9 (1977), pp. 1794–1797.

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Goal 1: Existence of solitary wave solutions for (NLS)

Opposed to blow-up: *solitary waves* of the form

$$\psi(t,x) = \mathrm{e}^{\boldsymbol{i}t}Q(x)$$

where $Q \in H^1(\mathbb{R}^N;\mathbb{R}) = H^1(\mathbb{R}^N)$ is a distributional solution of the nonlinear elliptic equation

$$-\Delta Q + Q = |Q|^{q-2}Q. \tag{PDE_Q}$$

Goal 2: Equality case in the Gagliardo-Nirenberg inequality

Theorem (Gagliardo-Nirenberg inequality)

For all $q \in]2, 2^*[$, *there exists a constant* $C(q) > 0$ *such that for every function* $v \in H^1(\mathbb{R}^N; \mathbb{C})$, *we have*

$$\|u\|_{L^q} \leq C(q)\, \|u\|_{L^2}^{1-s}\, \|\nabla u\|_{L^2}^{s}$$

where $s := \frac{(q-2)N}{2q}$.

Inequality + conservation laws ⟶ non-explosion criteria.

Optimal constant $C(q)$ ⟶ best criteria;

Passing to the modulus ⟶ only considering $u \geq 0$ is enough.

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Variational formulation

Gagliardo-Nirenberg inequality:

$$\|u\|_{L^q} \leq C(q) \|u\|_{L^2}^{1-s} \|\nabla u\|_{L^2}^{s}.$$

Goal: minimize the functional

$$\mathcal{J}(u) := \frac{\|u\|_{L^2}^{q(1-s)} \|\nabla u\|_{L^2}^{qs}}{\|u\|_{L^q}^{q}}$$

on $H^1(\mathbb{R}^N) \setminus \{0\}$.

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Link between the two goals (Existence of solitary wave solutions for (NLS) / Equality case for the Gagliardo-Nirenberg inequality)

The functional $\mathcal{J}$ is of class $\mathcal{C}^1$ on $H^1(\mathbb{R}^N) \setminus \{0\}$ and its differential is given by

$$\mathrm{d}\mathcal{J}(u) \cdot h = \mathcal{J}(u) \left(\frac{q(1-s)}{\|u\|_{L^2}^2} \int_{\mathbb{R}^N} u(x)h(x)\,\mathrm{d}x + \frac{qs}{\|\nabla u\|_{L^2}^2} \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla h(x)\,\mathrm{d}x - \frac{q}{\|u\|_{L^q}^q} \int_{\mathbb{R}^N} |u(x)|^{q-2} u(x) h(x)\,\mathrm{d}x \right)$$

for every $h \in H^1(\mathbb{R}^N)$. If u is a critical point of $\mathcal{J}$, we have

$$-\Delta u + \frac{(1-s)\|\nabla u\|_{L^2}^2}{s\|u\|_{L^2}^2} u = \frac{\|\nabla u\|_{L^2}^2}{s\|u\|_{L^q}^q} |u|^{q-2} u.$$

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Invariances of the functional

The functional

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where $s := \frac{(q-2)N}{2q}$ is invariant by:

- translations $u(x) \mapsto u(x - x_0)$ $(x_0 \in \mathbb{R}^N)$;
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Direct method from calculus of variations

Let's consider a minimizing sequence $(u_n)_{n\geq 1} \subseteq H^1(\mathbb{R}^N) \setminus \{0\}$, i.e. such that

$$\mathcal{J}(u_n) \xrightarrow[n\to\infty]{} \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \mathcal{J}(u).$$

We would like to extract a subsequence of $(u_n)_{n\geq 1}$ converging (weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q(\mathbb{R}^N)$) to a function $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and show that u is a minimum of $\mathcal{J}$.

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Compactness

Problem: loss of compactness by translations. If u is a global minimum of $\mathcal{J}$ and if $\xi \in \mathbb{R}^N \setminus \{0\}$, then the sequence of translates

$$\big(u(x - n\xi)\big)_{n\geq 1}$$

is a sequence of indistinguishable minima. If does not admit any strongly convergent subsequence in $L^q(\mathbb{R}^N)$.

Solution: work on the space $H^1_{\mathrm{r}}(\mathbb{R}^N)$ of $H^1(\mathbb{R}^N)$ *radial* functions.

Theorem (W. Strauss 1977)

If $N \geq 2$, the embedding of $H^1_{\mathrm{r}}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is compact for every $p \in \,]2, 2^*[$.

W. A. Strauss. "Existence of solitary waves in higher dimensions". In: *Comm. Math. Phys.* 55.2 (1977), pp. 149–162.

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If $N \geq 2$, the embedding of $H^1_{\mathrm{r}}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is compact for every $p \in]2, 2^*[$.

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Compactness

Problem: loss of compactness by translations. If u is a global minimum of $\mathcal{J}$ and if $\xi \in \mathbb{R}^N \setminus \{0\}$, then the sequence of translates

$$\big(u(x - n\xi)\big)_{n\geq 1}$$

is a sequence of indistinguishable minima. If does not admit any strongly convergent subsequence in $L^q(\mathbb{R}^N)$.

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Reduction to the nonnegative radial case

Data: minimizing sequence $(u_n)_{n\geq 1} \subseteq H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$\mathcal{J}(u_n) \xrightarrow[n\to\infty]{} \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \mathcal{J}(u).$$

- By passing to the absolute value, we can suppose that $u_n \geq 0$. We can thus work in

$$H^1_+(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, u \geq 0 \right\}.$$

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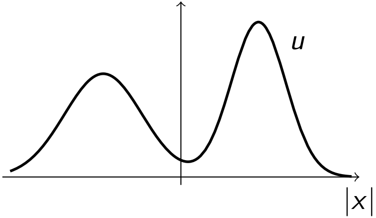
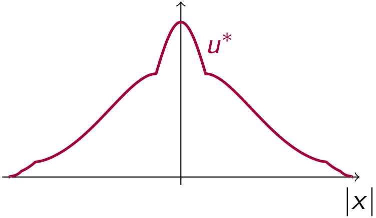
Symmetric decreasing rearrangement

Given a positive function $u : \mathbb{R}^N \to [0, +\infty]$, we consider its superlevel sets

$$\{x \in \mathbb{R}^N \mid u(x) > t\}$$

and we symmetrize them in an open ball centered in 0 with the same volume.





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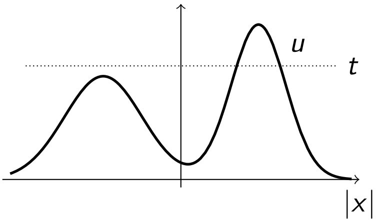

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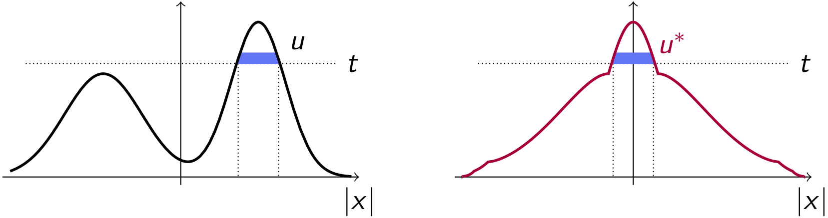

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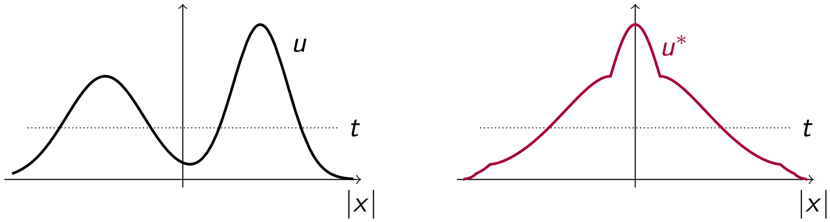

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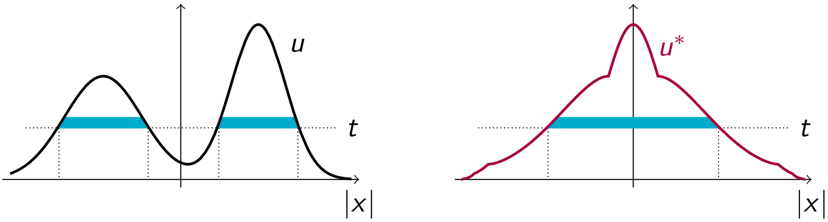


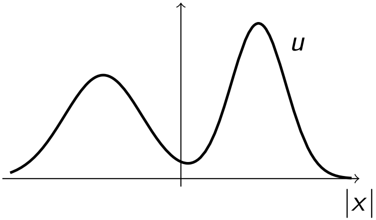
Rearrangement in H^1_+

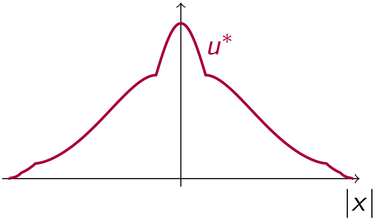
Theorem (Conservation of L^2 norms, Pólya–Szegő inequality)

If $u \in H^1_+(\mathbb{R}^N)$, then u^* also belongs to $H^1_+(\mathbb{R}^N)$ and we have

$$\|u^*\|_{L^2} = \|u\|_{L^2},$$

$$\|\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}.$$





Conclusion:
Existence of a radial positive minimum of $\mathcal{J}$

Steps:

$$(u_n)_{n\geq 1} \longrightarrow (|u_n|)_{n\geq 1} \longrightarrow (|u_n|^*)_{n\geq 1} \longrightarrow \text{compacity of the embedding}$$

Theorem (M.I. Weinstein 1982)

The equation

$$-\Delta Q + Q = |Q|^{q-2}Q \qquad (\mathrm{PDE}_Q)$$

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For every $k \geq 0$, there exists a radial sign-changing solution $Q_k(x) = u_k(|x|) \in H^1(\mathbb{R}^N)$ such that $[0, +\infty[\to \mathbb{R} : t \mapsto u_k(t)$ has exactly k roots.

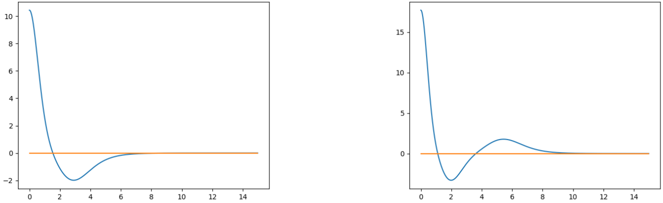

Figure: Graphs of u_1 and u_2 for $N = 3$ and $q = 3$

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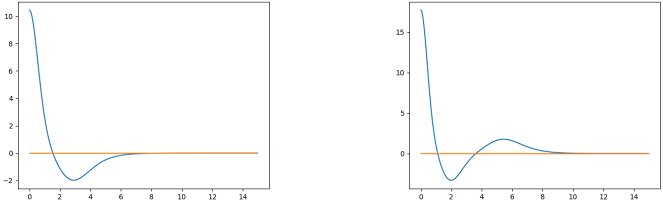

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If $N = 4$ or $N \geq 6$, then (PDE_Q) has a nonradial solution.

The main strategy consists in constructing (using variational methods) solutions based on another type of symmetry using the group

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There exist infinitely many $H^1(\mathbb{R}^2;\mathbb{R})$ solutions of

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whose maximal group of symmetry reduces to the identity.

The very rough idea is to start with an approximate solution of the form

$$S_{\text{approx}} = \sum_{z\in Z^+} Q(\cdot - z) - \sum_{z'\in Z^-} Q(\cdot - z')$$

for some *well-chosen* finite sets of points $Z^+, Z^- \subset \mathbb{R}^2$.

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Qualitative properties (of *all* $H^1(\mathbb{R}^N)$ solutions of (PDE_Q))

- elliptic bootstrap $\to$ all $H^1(\mathbb{R}^N)$ solutions of (PDE_Q) are $\mathcal{C}^2(\mathbb{R}^N)$;
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Energy and Pohožaev identites

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Energy of solitary wave solutions

The solutions $\tilde{Q} \in H^1(\mathbb{R}^N)$ to (PDE_Q) correspond to solitary wave solutions

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Therefore, solitary waves have a negative/zero/positive energy depending on whether $q < 2 + \frac{4}{N}$, $q = 2 + \frac{4}{N}$ or $q > 2 + \frac{4}{N}$;

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where $\lambda = N - 1$ and $t = |x|$.

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Studying radial solutions using ODEs

$\mathcal{C}^2$ *radial* solutions of (PDE_Q) correspond to solutions of the following Cauchy problem:

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Interpretation: dynamics of a nonlinear damped oscillator

Potential:

$$V(u) := \frac{|u|^q}{q} - \frac{|u|^2}{2}.$$

ODE:

$$\partial_{tt} u_y + \frac{\lambda}{t}\partial_t u_y + V'(u_y(t)) = 0.$$

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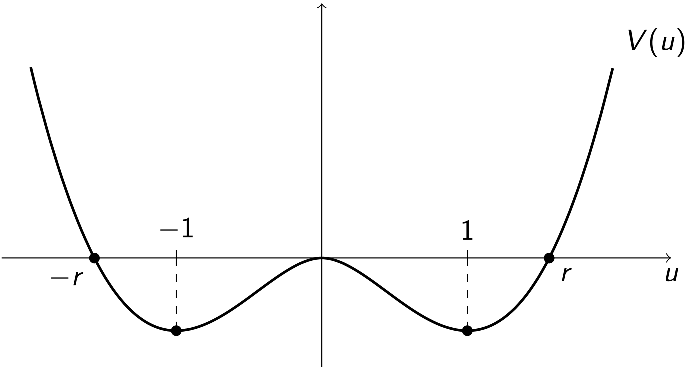
The potential well



$r = (q/2)^{\frac{1}{q-2}} > 1$

Behavior of solutions as $t \to +\infty$

Energy (*unrelated* to the energy of NLS as $t = |x|$ in the ODE setting):

$$H(u_y(t), \partial_t u_y(t)) = \frac{1}{2}|\partial_t u_y(t)|^2 + V(u_y(t))$$

Damping:

$$\partial_t \Big(t \mapsto H(u_y(t), \partial_t u_y(t)) \Big) = -\frac{\lambda}{t}|\partial_t u_y(t)|^2 \leq 0$$

Theorem

Every solution of (ODE_u) *converges to* -1, 0 *or* 1 *as* $t \to +\infty$.

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Shooting method: illustration

See the blackboard and animations!

Used parameters:

$$\lambda = 1, \qquad q = 2{,}5.$$

Uniqueness of the ground state: history

Theorem

There exists a unique $y > 0$ such that the associated solution of (ODE_u) *(with $u(0) = y$) is a "ground state solution", i.e.*

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Uniqueness: what about nodal solutions?

Conjecture

For every $k \in \mathbb{N}$, there exists a unique initial condition $y_k > 0$ such that the associated solution $u_{y_k}(t)$ has exactly k roots and converges to 0 as $t \to +\infty$.

Open for most values of q and λ, even for $k = 1$.

Recent computer-assisted proof (for fixed k, q and $\lambda = N - 1$):

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Back to the Gagliardo-Nirenberg inequality

Uniqueness of positive solutions to (PDE_Q) allows to characterize all equality cases in the Galigardo-Nirenberg inequality.

Theorem (Equality cases in the Gagliardo-Nirenberg inequality)

The global minima on $H^1(\mathbb{R}^N) \setminus \{0\}$ of functional

$$\mathcal{J}(u) := \frac{\|u\|_{L^2}^{q(1-s)} \, \|\nabla u\|_{L^2}^{qs}}{\|u\|_{L^q}^{q}},$$

where $s := \frac{(q-2)N}{2q}$, are the functions of the form

$$u(x) = \mu Q(\lambda(x - x_0))$$

where $\mu \in \mathbb{R} \setminus \{0\}$, $\lambda > 0$ and $x_0 \in \mathbb{R}^N$.

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Non-explosion criteria

Since Q is a global minimum of $\mathcal{J}$, we obtain that

$$\frac{2\|Q\|_{L^2}^{q-2}}{q} = \mathcal{J}(Q) \leq \mathcal{J}(u) = \frac{\|u\|_{L^2}^{q(1-s)} \, \|\nabla u\|_{L^2}^{qs}}{\|u\|_{L^q}^{q}}$$

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Conservation laws and the Gagliardo-Nirenberg inequality with optimal constant $\mathcal{J}(Q)$ imply that, for all $t \in [0, T_{\max}[$,

$$\|\nabla \psi(t,\cdot)\|_{L^2}^2 \leq 2\mathcal{E}(\psi_0) + \frac{2}{q}\|\psi(t,\cdot)\|_{L^q}^q$$

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Non-explosion below the mass-critical exponent

For all $t \in [0, T_{\max}[$, we obtained the bound

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If $q < 2 + \frac{4}{N}$, then $qs < 2$ (since $s := \frac{(q-2)N}{2q}$), so we obtain a uniform bound in t for $\|\nabla\psi(t,\cdot)\|_{L^2}^2$, and there is no blow-up.

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Many features of the mass-critical exponent

- Glassey's argument applies iff $q \geq 2 + \frac{4}{N}$;
- Solitary waves have a negative/zero/positive energy depending on whether $q < 2 + \frac{4}{N}$, $q = 2 + \frac{4}{N}$ or $q > 2 + \frac{4}{N}$;
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- When $q = 2 + \frac{4}{N}$, (NLS) enjoys an extra *pseudo-conformal* symmetry. If $\psi(t,x)$ solves (NLS) for $q = 2 + \frac{4}{N}$, so does

$$\left(\frac{T}{T-t}\right)^{\frac{N}{2}} \psi\left(\frac{tT}{T-t}, \frac{xT}{T-t}\right) e^{-i\frac{|x|^2}{4(T-t)}}.$$

From now on, we consider the mass-critical case $q = 2 + \frac{4}{N}$.

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Blow-up thresholds in the mass-critical case

If $q = 2 + \frac{4}{N}$, we can rewrite the bound

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Existence of minimal mass blow-up solutions

If $\|\psi_0\|_{L^2} = \|Q\|_{L^2}$, blow-up is possible. The explicit solution

$$s_T(t,x) := \left(\frac{T}{T-t}\right)^{N/2} Q\left(\frac{xT}{T-t}\right) \exp\left(\boldsymbol{i}\left(\frac{Tt}{T-t} - \frac{|x|^2}{4(T-t)}\right)\right) \tag{1}$$

obtained by the pseudo-conformal transform blows up at time $t = T$.

Remark

The complex exponential is very important. Indeed, for all $x \in \mathbb{R}^N$,

$$|s_T(0,x)| = |Q(x)|,$$

but the initial condition $\psi_0 = Q$ gives rise to the solitary wave solution $e^{\boldsymbol{i}t}Q(x)$, which does not blow-up!

It turns out that solutions of the form (1) are the only minimal mass solutions of (NLS) when $q = 2 + \frac{4}{N}$, up to the symmetries of the equation.

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Classification of minimal mass solutions

Theorem (F. Merle 1993)

If $\psi(t,x)$ is a solution of (NLS), defined for $t \in [0,T[$ and blowing up for $t = T$, then there exist $\theta \in \mathbb{R}, \omega \in]0,+\infty[, x_0 \in \mathbb{R}^N, x_1 \in \mathbb{R}^N$ such that

$$\psi_0 = \left(\frac{\omega}{T}\right)^{N/2} e^{i\theta - i|x-x_1|/4T + i\omega^2/T} Q\left(\omega\left(\frac{x-x_1}{T} - x_0\right)\right).$$

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Classification of minimal mass solutions

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Further study of $s_T(t,x)$

If

$$s_T(t,x) := \left(\tfrac{T}{T-t}\right)^{N/2} Q\left(\tfrac{xT}{T-t}\right) \exp\left(\boldsymbol{i}\left(\tfrac{Tt}{T-t} - \tfrac{|x|^2}{4(T-t)}\right)\right),$$

then the variance of $|s_T(t,\cdot)|^2$ is given by

$$\begin{aligned} V(t) &= \int_{\mathbb{R}^N} |x|^2 |s_T(t,x)|^2 \,\mathrm{d}x \\ &= \left(\frac{T}{T-t}\right)^N \int_{\mathbb{R}^N} |x|^2 Q\left(\frac{xT}{T-t}\right)^2 \mathrm{d}x \\ &= \left(\frac{T-t}{T}\right)^2 V(0) \\ &\xrightarrow[t\to T]{} 0 \end{aligned}$$

The variance identity implies that

$$\partial_{tt} V(t) = \frac{2}{T^2} V(0) = 16\mathcal{E}(s_T(t,\cdot))$$

Further study of $s_T(t,x)$

- The pseudo-conformal solutions have a strictly positive energy;
- The variance of $s_T(t,\cdot)$ converges to 0 as $t \to T$;
- For all $t \in [0,T[$, we have

$$\|s_T(t,\cdot)\|_{L^2} = \|Q\|_{L^2}.$$

- The two previous points imply that

$$|s_T(t,\cdot)|^2 \xrightarrow[t\to T]{\mathcal{S}'(\mathbb{R}^N)} |Q|_{L^2}^2 \delta_0.$$

- Blow-up rate:

$$|\nabla s_T(t,\cdot)|_{L^2} = \frac{T\|\nabla Q\|_{L^2}}{T-t}$$

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Bourgain-Wang solutions

Question: what happens if $\|\psi_0\|_{L^2} > \|Q\|_{L^2}$?

Theorem (J. Bourgain, W. Wang 1997)

If $N = 1$ or $N = 2$, the mass-critical (NLS) equation admits solutions $\psi(t,x) \in \mathcal{C}([0,T[, H^1(\mathbb{R}^N;\mathbb{C}))$ with $\|\psi(t,\cdot)\|_{L^2} > \|Q\|_{L^2}$ blowing up at time $T > 0$ at the rate

$$\|\psi(t,\cdot)\|_{L^2} \sim \frac{C}{T-t}$$

near blow-up time.

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The log-log blow-up rate

- s_T solutions have a strictly positive energy, while Glassey's argument shows that there are many solutions with a negative energy;
- Solutions blowing up with rate $\frac{C}{T-t}$ are not observed in numerical simulations;
- In the 1980s, it was suspected that the log-log law

$$\|\psi(t,\cdot)\|_{L^2} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{1/2}$$

was the generic blow-up speed.

Historical context: see e.g.

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Results of Frank Merle and Pierre Raphaël (context)

- The following results will assume $N = 1$ or $N \geq 2$ and a certain "spectral property" holds true (see later).
- They concern the mass-critical case $q = 2 + \frac{4}{N}$.
- We will consider initial profiles $\psi_0 \in H^1(\mathbb{R}^N; \mathbb{C})$ satisfying

$$\|Q\|_{L^2}^2 \leq \|\psi_0\|_{L^2}^2 \leq \|Q\|_{L^2}^2 + \alpha^*. \qquad (2)$$

  For all N, the following theorems will provide the existence of a suitable $\alpha^* > 0$ such that the conclusions of the theorems hold for all $\psi_0 \in H^1(\mathbb{R}^N; \mathbb{C})$ such that (2) holds.
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Description of the singularity

Theorem

Assume that $u(t)$ blows up in finite time, i.e. $T_{\max} < +\infty$. Then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in]0, +\infty[\times \mathbb{R}^N \times \mathbb{R}$ and an asymptotic profile $u^* \in L^2(\mathbb{R}^N)$ such that

$$\psi(t,\cdot) - \frac{1}{\lambda(t)^{N/2}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \xrightarrow[t \to T]{L^2} u^*.$$

Moreover, the blow-up point is finite in the sense that

$$x(t) \xrightarrow[t \to T]{} x(T) \in \mathbb{R}^N.$$

Estimates on the blow up speed

Theorem

We have either

$$\frac{\|\nabla\psi(t,\cdot)\|_{L^2}}{\|\nabla Q\|_{L^2}}\left(\frac{T-t}{\log|\log(T-t)|}\right)^{1/2}\xrightarrow[t\to T]{}\frac{1}{\sqrt{2\pi}},$$

or

$$\|\nabla\psi(t,\cdot)\|_{L^2}\geq\frac{C(\psi_0)}{T-t},$$

as $t\to T$.

Sufficient condition for log-log blow-up, stability of the rate

Theorem

If $\mathcal{E}(u_0) \leq 0$ and $\|\psi_0\|_{L^2} > \|Q\|_{L^2}$, then $\psi(t,\cdot)$ blows up in finite time with the log-log speed.

Moreover, the set of initial profiles $\psi_0 \in H^1(\mathbb{R}^N)$ such that

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The spectral property (sketch)

The main concern is related to understanding what are the eigenvalues and eigenvectors of the two real Schrödinger operators

$$L_1 = -\Delta + V_1, \qquad L_1 = -\Delta + V_2,$$

where, still using the convention $t = |x|$ in the radial setting,

$$V_1(t) = \frac{2}{N}\left(\frac{4}{N} + 1\right) Q^{\frac{4}{N}-1} t\partial_t Q, \qquad V_2(t) = \frac{2}{N} Q^{\frac{4}{N}-1} t\partial_t Q.$$

In practice, we need to consider the ODE

$$\begin{cases} -\partial_{tt} U_i(t) - \frac{N-1}{t}\partial_t U_i(t) + V_i(t) U_i(t) = 0 \\ U_i(0) = 1, \quad \partial_t U_i(0) = 0, \end{cases}$$

and counting the number of zeros of U_i, when $i = 1$ and $i = 2$.

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F. Merle and P. Raphaël. “On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation”. In: *Invent. Math.* 156.3 (2004), pp. 565–672.

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and many more! For overviews, see

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Towards a computer-assisted proof of the spectral property

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Thanks for your attention!

Main used references (good starting points into NLS!)

- T. Tao. *Nonlinear dispersive equations*. Vol. 106. CBMS Regional Conference Series in Mathematics. Local and global analysis. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, pp. xvi+373.
- T. Cazenave. *An overview of the nonlinear Schrödinger equation*. Nov. 2020.
- T. Cazenave. *Semilinear Schrödinger equations*. Vol. 10. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003, pp. xiv+323.